

Higher abelian gauge theory associated to gerbes on noncommutative deformed M5-branes and S-duality

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Abstract

We enhance the action of higher abelian gauge theory associated to a gerbe on an M5-brane with an action of a torus \mathbb{T}^n ($n \geq 2$), by a noncommutative \mathbb{T}^n -deformation of the M5-brane. The ingredients of the noncommutative action and equations of motion include the deformed Hodge duality, deformed wedge product, and the noncommutative integral over the noncommutative space obtained by strict deformation quantization. As an application we then introduce a variant model with an enhanced action in which we show that the corresponding partition function is a modular form, which is a purely noncommutative geometry phenomenon since the usual theory only has a \mathbb{Z}_2 -symmetry. In particular, S-duality in this 6-dimensional higher abelian gauge theory model is shown to be, in this sense, on par with the usual 4-dimensional case.

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1 Introduction

The presence of a B-field in string theory generally makes the underlying space noncommutative (see [35]). This can appear on the worldvolume theory of branes as well as in space-time. D-branes support Yang-Mills fields, so the presence of the B-field leads to noncommutative Yang-Mills theory (see [10]). Recently there has been a lot of interest in the theory on the worldvolume of the M-theory fivebrane (M5-brane). This is a superconformal field theory (in some limit) which has a B-field in its field content (see e.g. [3] for a survey). For a special case of the worldvolume theory of the fivebrane, the ADHM construction can be extended to describe (effectively 4-dimensional) noncommutative instantons [29]. We will generalize an aspect of this to consider the full six-dimensional theory without relying on the reduction to Yang-Mills theory, although we will also consider interesting instances of such a reduction. As the fivebrane theory is believed to be intrinsically quantum, a description in terms of noncommutative geometry would be appropriate. Indeed, this has been considered previously in [2] using the C-field on the worldvolume to describe noncommutative fivebranes via open membranes. We will instead make use of the notion of strict deformation quantization of Rieffel [31]. For a parametrized version of strict deformation quantization and its applications to T-duality, see [20].

The theory on the worldvolume of a single fivebrane, given by a closed oriented 6-dimensional manifold, can be described as an abelian gerbe theory (cf. [5, 6, 28] for extensive description of gerbes). On the other hand, an action principle for gerbes was studied in [24]

$$S(B) = \int_{M^6} H_B \wedge *H_B , \quad (1.1)$$

where B is the B-field of the gerbe whose 3-curvature is $H_B = dB$ with Dixmier-Douady class $[H_B] \in H^3(M; \mathbb{Z})$. This theory only has a \mathbb{Z}_2 -symmetry given by $B \rightarrow -B$. Such

an action at the level of differential forms also appears in the literature as describing part of the dynamics of the fivebrane, but would vanish upon imposing the desired self-duality equations. We will take this as the starting point to propose, with a remedy, an action for the fivebrane including noncommutativity and self-duality. Classically, the equations describing the dynamics of the H-field are

$$d^\dagger H_B = 0, \quad dH_B = 0, \quad (1.2)$$

where $*$ is the Hodge duality operator in six dimensions, d is the exterior derivative on forms, and d^\dagger is the adjoint of d . The solutions are the harmonic 3-forms on the M5-brane M^6 . One can also add source terms and also impose self-duality by hand, in which case the action (1.1) might be referred to as a pseudo-action (see [1]). Recent accounts of higher abelian gauge theory in this context, via differential cohomology, are given in [15, 16, 17, 36].

Note that in our setting the source of commutativity will not be a physical parameter like the C-field, but rather a mathematical parameter that we introduce into the theory. As such, from the physical point of view our setting describes a six-dimensional model of abelian gerbe theory associated to the M5-brane. At this stage it might benefit the reader to dissect the title of this paper: The model that we study is the “Higher abelian gauge theory” and we have explained above that it is one that essentially appears on the worldvolume of M5-branes. Furthermore, we consider the underlying M5-brane worldvolumes to be “noncommutative deformed” by adding a parameter θ via Rieffel’s strict deformation quantization. Aside from studying the model for its own right, we also discover that there are certain desirable byproducts of doing so. The first is that one can study “S-duality” in this setting in essentially a similar way that is done for usual S-duality in four dimensions. The latter has far-reaching consequences on geometric topology and the Geometric Langlands Program [41, 42]. We hope that this paper is a first step in extending these connections directly to six dimensions.

The main content. We provide the following fix to the action whenever the M5-brane M^6 has an action of a torus \mathbb{T}^n for some $n \geq 2$. To give the action and the partition function (of a variant model) we describe a noncommutative \mathbb{T}^n -deformation M_θ^6 of the M5-brane worldvolume, of the integral, of the Hodge star operator, and of the wedge product. We propose the new action, where we will assume that 3-curvature is self-adjoint $H_B^\dagger = H_B$ with respect to the deformed inner product,

$$S_\theta(B) = \int_{M_\theta} H_B \wedge_\theta *_\theta H_B - \int_{M_\theta^6} H_B \wedge_\theta H_B + \int_{M_\theta^6} C_6, \quad (1.3)$$

and where C_6 is a potential for the dual of the C-field in M-theory pulled back to the fivebrane worldvolume.

The equations of motion (EOM) and Bianchi identity³ of the H-field can be derived as in the case of noncommutative Yang-Mills [23] to be

$$d^{\dagger_\theta} H_B = 0, \quad dH_B = 0, \quad (1.4)$$

³For brevity, we will refer to both as EOMs.

where $d^{\dagger\theta}$ is the adjoint of d with respect to the deformed inner product

$$(H, H')_{\theta} = \int_{M_{\theta}^6} H^{\dagger} \wedge_{\theta} *_{\theta} H'.$$

Since $d^{\dagger\theta} = \pm *_{\theta} d *_{\theta}$ (cf. [23]), the EOM in (1.4) can be re-written as

$$d *_{\theta} H_B = 0, \quad dH_B = 0. \quad (1.5)$$

1. The main goal is to show that the above action makes sense by describing the following ingredients in the next section,

- (a) the noncommutative space M_{θ}^6 , as a deformation of the worldvolume M^6 ,
- (b) the operators corresponding to H_3 and C_6 at the quantum level,
- (c) the deformed wedge product \wedge_{θ} ,
- (d) the deformed Hodge star operator $*_{\theta}$, and
- (e) the noncommutative integral $\int_{M_{\theta}^6}$.

We hope these will be of independent interest for other settings as well. Such deformations have the virtue of keeping key properties of the underlying classical geometry unaltered, and as such many classical concepts can be extended through the deformation in this approach.

2. The second goal is setting up the partition function, which we do in Sec. 3. In the commutative case, self-duality of $(2p+1)$ -forms in $4p+2$ dimensions poses a problem as far as an action principle goes. Naively, the kinetic term would vanish identically upon imposing self-duality, as the action would then be the wedge product of an odd degree differential form with itself. Consequently, this then implies a problem for the partition function. We would like to warn the reader that the fivebrane partition function is a complicated issue due to the self-duality constraint and that the usual method of dealing with it is via holomorphic factorization, which leads to an ambiguity in its definition and is related to Spin structures on the worldvolume (see [39] for a detailed explanation). There have been proposals to evade this by not working directly in $4p+2$ dimensions but rather extending to a Chern-Simons theory in $4p+3$ dimensions and/or to the bounding theory in $4p+4$ dimensions. This allows the partition function to be defined as a section of a line bundle over the intermediate Jacobian, and requires a quadratic refinement [39]. Discussions on extension to (higher) differential cohomology and stacks are given in [15, 16, 17]. The formulation that we propose via noncommutative geometry does not suffer from such an immediate problem, because ultimately $H_{2p+1} \wedge_{\theta} H_{2p+1} \neq 0$. Therefore, for $p=1$, the noncommutative deformation removes this subtlety or difficulty that has plagued the study of the M5-brane.⁴

3. The third goal is to calculate the partition function of the model. For this we follow the discussion of the 4-dimensional abelian Yang-Mills case to evaluate the partition function

⁴An analogous argument extends to the other physically important cases, namely the self-dual scalar in $d=2$ and to type IIB string theory in $d=10$, i.e. for $p=0$ and $p=2$, respectively.

via functional determinants. This involves summing over gerbe connections and requires an extension of some analytic concepts, such as Hodge theory, from the classical to the noncommutative (in our sense) setting. This leads to determinants of Laplace-type operators, which we interpret via Faddeev-Popov ghosts as well as ghost-for-ghost determinants. A regularization of the determinants is needed, and we choose the ζ -function regularization. Extension to the global case when the gerbe is not trivial requires summing over the moduli spaces of solutions to the gerbe curvature equations. We also account for torsion in the curvature, which extends the moduli space of solutions (instantons).

4. The fourth goal, which can be viewed as an application of the above construction, is to provide a variant model in which we can study a form of S-duality. This model, in which we explore how much certain aspects of the 4-dimensional undeformed theory can be carried over to the 6-dimensional deformed theory, will not fully capture the dynamics of the M5-brane theory, but we hope it will nevertheless give some insight into that theory. We modify the action (1.3) to include coupling parameters ⁵

$$S_\theta(B; \tau) = \frac{1}{2e^2} \int_{M_\theta^6} H_B \wedge *_\theta H_B + \frac{i\Theta}{2} \int_{M_\theta^6} H_B \wedge_\theta H_B + \int_{M_\theta^6} C_6, \quad (1.6)$$

with coupling parameter $\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{e^2}$. The main point to highlight is that the deformed wedge product is no longer skew-symmetric, so that now it is possible to restore the full $SL(2, \mathbb{Z})$ symmetry to the partition function associated to the action $S_\theta(B; \tau)$ and, furthermore, that it is a purely noncommutative geometry phenomenon, similar in spirit to the renowned work by Nekrasov and Schwartz [29] in another context as alluded to earlier.

We also study the partition function $Z_\theta(M_\theta^6; \tau)$ associated to the action $S_\theta(B; \tau)$ and show that it is a modular form, and identify the modular weights. Furthermore, under some assumptions on heat kernels in the noncommutative setting, we also discuss curvature corrections to these modular weights. A full account would require an extension of the seminal work of Vafa-Witten [37] to our setting, which goes far beyond the scope of the current paper. Nevertheless, we expect this fact to have important consequences somewhat similar to the far-reaching 4-dimensional case. Traditionally, modularity in four dimensions is explained by dimensional reduction on the 2-torus from a six dimensional theory, which generally does not possess any modularity. What we do here uncovers modularity already in six dimensions, a striking phenomenon that is a result of noncommutativity there.

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⁵We emphasize the fact that this where we depart from the M5-brane theory, which does not have adjustable coupling parameters. We thank David Berman for remarks on this point.

2 The noncommutative setting for the fivebrane

In this section we provide the main construction and proposal of this letter, which is to enhance aspects of the fivebrane worldvolume theory to the noncommutative setting via strict deformation quantization. The virtue of this approach is that the ingredients and calculations are relatively transparent hence utilizable in calculations and, furthermore, can be adapted to other settings. A good part of the ensuing discussion is an adaptation to our setting of known constructions in noncommutative geometry, but there are also new constructions and definitions; in particular, the noncommutative wedge product in Sec. 2.3.

2.1 Noncommutative worldvolumes

Let M^6 be a compact Riemannian Spin manifold (without boundary) of dimension six whose isometry group has rank $r \geq 2$. Then M^6 admits natural isospectral deformations to noncommutative geometries M_θ , with an antisymmetric deformation parameter $\theta = (\theta_{ab} = -\theta_{ba})$, $\theta_{ab} \in \mathbb{R}$; see [9, 8, 23], which we follow in this section. The idea is to deform the standard spectral triple describing the Riemannian geometry of M^6 along a torus embedded in the isometry group to get an isospectral triple $(C^\infty(M_\theta), \mathcal{H}, \mathcal{D}, \gamma)$, where \mathcal{H} is the Hilbert space, D is the Dirac operator, and γ is the chirality operator [8]. This is done by deforming the torus action.

The natural one-parameter deformation can be taken to be isospectral, i.e. leaving the Dirac operator D unchanged, and the algebra of smooth functions $C^\infty(M_\theta^6)$ in the noncommutative geometry M_θ^6 can be described in terms of the quantization of smooth functions $L_\theta(C^\infty(M^6))$ on the underlying classical geometry M^6 . The noncommutative Spin geometry will be $(L_\theta(C^\infty(M)), \mathcal{H}, \mathcal{D})$. Note that in this approach all spectral properties are preserved.

Consider the isometric smooth action σ of \mathbb{T}^n , $2 \leq n \leq 6$, on M^6 . Decompose the classical algebra of smooth functions $C^\infty(M^6)$ into spectral subspaces indexed by the dual group $\mathbb{Z}^n = \widehat{\mathbb{T}^n}$: each $r \in \mathbb{Z}^n$ labels a character of \mathbb{T}^n via $e^{2\pi i s} \mapsto e^{2\pi i r \cdot s}$. The r -th spectral subspace for σ on $C^\infty(M^6)$ is formed of functions f_r such that $\sigma_s(f_r) = e^{2\pi i r \cdot s} f_r$, each $f \in C^\infty(M^6)$ is the sum of a unique (rapidly convergent) series $f = \sum_{r \in \mathbb{Z}^n} f_r$. With $\theta = (\theta_{jk} = -\theta_{kj})$ a real antisymmetric $n \times n$ matrix, replace the ordinary product by a deformed product $f_r \times_\theta g_{r'} := f_r \sigma_{\frac{1}{2}r \cdot \theta}(g_{r'}) = e^{\pi i r \cdot \theta \cdot r'} f_r g_{r'}$, and denote $C^\infty(M_\theta^6) := (C^\infty(M^6), \times_\theta)$. The action σ of \mathbb{T}^n extends to $C^\infty(M_\theta^6)$. At the level of the C^* -algebra of continuous functions one has a strict deformation quantization in the direction of the Poisson structure defined by the matrix θ . The quantization of smooth functions is given by the *quantization map*

$$L_\theta : C^\infty(M^6) \rightarrow C^\infty(M_\theta^6), \quad (2.1)$$

which satisfies $L_\theta(f \times_\theta g) = L_\theta(f)L_\theta(g)$. See [9, 8, 23] for more details.

2.2 The noncommutative integral

Corresponding to the spectral triple $(C^\infty(M_\theta), \mathcal{H}, D)$ is a noncommutative integral defined as a Dixmier trace (see [7])

$$\oint L_\theta(f) := \text{Tr}_\omega(L_\theta(f)|D|^{-6}) \quad (2.2)$$

with $f \in C^\infty(M^6)$ via its representation on the Hilbert space \mathcal{H} .

The C_6 -integral. We will use the following [18] [23] as the definition of the volume form on M_θ^6

$$\oint L_\theta(f) = \int_{M^6} f d\nu . \quad (2.3)$$

The integral over M_θ^6 can be defined using the quantum integral of the operators corresponding to the differential forms. For $C_6 \in \Omega^6(M_\theta^6)$ we define

$$\int_{M_\theta^6} C_6 := \oint *_\theta C_6 , \quad (2.4)$$

where $*_\theta C_6$ is an element in $C^\infty(M_\theta^6)$ and the right-hand side is defined as in (2.3). If C_6 is an exact form, that is if $C_6 = dA_5$ for some 5-form $A_5 \in \Omega^5(M_\theta^6)$, then it can be checked that the integral of C_6 will be zero. This is consistent with – and is in a sense a quantum version of – the usual requirement of having fivebranes with no boundaries.

The other integrals in the action are defined in an analogous manner.

2.3 Deformed wedge product

Consider the action of \mathbb{T}^n , $n \geq 2$, on M^6 . This action induces an action of \mathbb{T}^n on the space of differential forms $\Omega(M^6)$ on M^6 . Starting with a $U(1)$ -cocycle $\theta \in Z^2(\widehat{\mathbb{T}^n}, U(1))$ on the dual to the torus $\widehat{\mathbb{T}^n}$, we would like to deform the wedge product. To that end, we decompose the space of differential forms with respect to the characters of the torus group

$$\Omega(M^6) \cong \bigoplus_{\alpha \in \widehat{\mathbb{T}^n}} \Omega(M^6)_\alpha , \quad (2.5)$$

where the components in the decomposition are given by

$$\Omega(M^6)_\alpha := \{ \omega \in \Omega(M^6) \mid t^*(\omega) = \alpha(t)\omega \text{ for all } t \in \mathbb{T}^n \} . \quad (2.6)$$

Correspondingly, we write a differential form in components as

$$\omega = \sum_{\alpha \in \widehat{\mathbb{T}^n}} \omega_\alpha , \quad \omega_\alpha \in \Omega(M^6)_\alpha . \quad (2.7)$$

Then the wedge product on components takes the form

$$(\omega \wedge \eta)_\alpha = \sum_{\alpha_1 + \alpha_2 = \alpha} (\omega_{\alpha_1} \wedge \eta_{\alpha_2}) . \quad (2.8)$$

We then define the components of the deformed wedge product \wedge_θ to be

$$(\omega \wedge_\theta \eta)_\alpha := \sum_{\alpha_1 + \alpha_2 = \alpha} \omega_{\alpha_1} \wedge \eta_{\alpha_2} \theta(\alpha_1, \alpha_2) . \quad (2.9)$$

The deformed wedge product \wedge_θ is no longer skew-symmetric in general, as we have that $\theta(\alpha_1, \alpha_2) = \overline{\theta(\alpha_2, \alpha_1)}$, i.e. θ is a phase.

The action of the de Rham differential on the deformed wedge product is given by

$$\begin{aligned} d(\omega \wedge_\theta \eta)_\alpha &= \sum_{\alpha_1 + \alpha_2 = \alpha} d(\omega_{\alpha_1} \wedge \eta_{\alpha_2}) \theta(\alpha_1, \alpha_2) \\ &= \sum_{\alpha_1 + \alpha_2 = \alpha} (d\omega_{\alpha_1} \wedge \eta_{\alpha_2} + (-1)^{\deg(\omega)} \omega_{\alpha_1} \wedge d\eta_{\alpha_2}) \theta(\alpha_1, \alpha_2) \\ &= (d\omega \wedge_\theta \eta)_\alpha + (-1)^{\deg(\omega)} (\omega \wedge_\theta d\eta)_\alpha , \end{aligned} \quad (2.10)$$

hence

$$d(\omega \wedge_\theta \eta) = d\omega \wedge_\theta \eta + (-1)^{\deg(\omega)} \omega \wedge_\theta d\eta . \quad (2.11)$$

Therefore the deformed wedge product \wedge_θ induces a product on de Rham cohomology ⁶ $H^\bullet(M^6)$. We will next compare this product structure with the product structure determined by the standard wedge product.

Let $[0, 1] \ni t \mapsto \theta_t \in Z^2(\widehat{\mathbb{T}}^n, U(1))$ be a 1-parameter family of cocycles. Then we get a homotopy $[0, 1] \ni t \mapsto \wedge_{\theta_t}$ of wedge products on cohomology, and a standard argument shows that $\wedge_{\theta_0} = \wedge_{\theta_1}$ on de Rham cohomology. Such homotopies are obtained by choosing $\xi \in Z^2(\widehat{\mathbb{T}}^n, \mathbb{R})$ and considering the homotopy $[0, 1] \ni t \mapsto \theta_t = \exp(2\pi i t \xi) \theta \in Z^2(\widehat{\mathbb{T}}^n, U(1))$. By considering the long exact sequence in cohomology associated to the exact sequence of coefficients

$$1 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1) \rightarrow 1$$

and, noting that the torsion subgroup of $H^3(\widehat{\mathbb{T}}^n, \mathbb{Z})$ is trivial, we conclude that the deformed wedge product induces the same product structure on de Rham cohomology as does the usual wedge product.

Define the *deformed algebra of differential forms* to be $\Omega^p(M_\theta^6) = (\Omega^p(M^6), \wedge_\theta)$.

⁶which still has the same classical definition since the differential and the space of forms have not changed.

2.4 Deformed Hodge star operator

We will, as before, consider the worldvolume M^6 with a Riemannian metric g and with an action of a torus \mathbb{T}^n by isometries. Considering isospectral deformations, in which the metric is unchanged, the deformed Hodge star operator

$$*_\theta : \Omega^p(M_\theta^6) \rightarrow \Omega^{6-p}(M_\theta^6) \quad (2.12)$$

is defined, as in [23], by the commutative diagram

$$\begin{array}{ccc} \Omega^p(M^6) & \xrightarrow{*} & \Omega^{6-p}(M^6) \\ \downarrow L_\theta & & \downarrow L_\theta \\ \Omega^p(M_\theta^6) & \xrightarrow{*_\theta} & \Omega^{6-p}(M_\theta^6) . \end{array} \quad (2.13)$$

That is, we define the deformed Hodge star operator on $H \in \Omega^p(M_\theta^6)$ by

$$*_\theta H = L_\theta * L_\theta^{-1}(H) . \quad (2.14)$$

This will be one of the ingredients in building the action and the equations of motion.

The inner product on $\Omega(M_\theta^6)$ and the kinetic action functional. The inner product, for two p -forms $H, H' \in \Omega^p(M_\theta^6)$, is

$$(H, H')_\theta = \int *_\theta(H^\dagger \wedge_\theta *_\theta H') \quad (2.15)$$

since $*_\theta(H^\dagger \wedge_\theta *_\theta H') \in C^\infty(M_\theta^6)$. Here H^\dagger is the adjoint operator corresponding to the operator form of H . We now consider how the noncommutative wedge product \wedge_θ and the noncommutative Hodge star operator $*_\theta$ work together. Using the above notions and definitions we build the expression that appears in the proposed action (1.6), that is

$$H_3 \wedge_\theta *_\theta H_3 . \quad (2.16)$$

An alternate expression for the inner product (2.15), is given by the noncommutative integral

$$(H, H')_\theta = \int_{M_\theta^6} H^\dagger \wedge_\theta *_\theta H' . \quad (2.17)$$

It is easy to see the following symmetries of the inner product

$$(H, H')_\theta = \overline{(H', H)_\theta} , \quad (aH, bH')_\theta = \bar{a}b(H, H')_\theta . \quad (2.18)$$

The nondegeneracy is explained in [4] for example.⁷ Either of the two expressions, (2.15) or (2.17), can be taken to be the kinetic term of the H-field. Note that for $H = H'$ this implies that the inner product is real. This is also a generalization/analog of the nonabelian Yang-Mills description in four dimensions in [23] to abelian gerbe theory in six dimensions.

⁷in the Yang-Mills case, but the formulation is general.

2.5 Torus bundles and strict deformation quantization

The fivebrane worldvolume theory can be considered on tori as well as on torus bundles. Different fiber dimensions capture different physical aspects of the theory. As an application of our construction, we discuss here the example the case of a principal 2-torus bundle, which is also physically relevant.

Consider fivebrane worldvolume M^6 as the total space of a principal 2-torus bundle over an oriented 4-dimensional manifold X^4 . As before, we strictly deform quantize with respect to the 2-torus action to get a noncommutative principal torus bundle (NCTP) with total space M_θ^6 and constant deformation parameter θ , cf. [20, 21]. NCTP bundles occur in the study of T-duality in a background flux [25, 26, 27] in string theory, and was first described in terms of strict deformation quantization in [20, 21].

We start by recalling the commutative case, from [38, 12, 41]. The classical Hodge star operator $* : \Omega^p(M^6) \rightarrow \Omega^{6-p}(M^6)$ depends only on the conformal class of the metric on M^6 . Classically, the equations

$$d^\dagger H_3 = \pm * d * H_3 = 0, \quad dH_3 = 0 \quad (2.19)$$

are therefore also conformally invariant. The passage to the quantum theory preserves this property since the theory is linear [41].

Consider the worldvolume as the product $M^6 = X^4 \times \mathbb{T}^2$ with the product conformal structure. The reduction of the theory on \mathbb{T}^2 results in an induced 4-dimensional theory which depends on the conformal structure of \mathbb{T}^2 up to isomorphism. The latter is determined by a choice of a point τ in the upper half plane modulo the action of the modular group $SL(2, \mathbb{Z})$. With complex coordinates $z = x + iy$ the ansatz for the H-field is

$$H_3 = F_2 \wedge dx + *_4 F_2 \wedge dy, \quad (2.20)$$

where F_2 is a two-form pulled back from X^4 to M^6 and $*_4$ is the Hodge star operator on X^4 . Classically, the Bianchi identity (or equation of motion with self-duality condition) $dH_3 = 0$ gives rise to Maxwell's equations $dF_2 = d *_4 F_2 = 0$.

We propose a description in terms of the noncommutative deformation. Equation (1.5) is then conformally invariant in the noncommutative sense. This is an important ingredient in viewing (a limit of) the theory as an (exotic) (2,0) conformal field theory. In the special case when M^6 is the trivial bundle $X^4 \times \mathbb{T}^2$, we have that M_θ^6 is just $X^4 \times \mathbb{T}_\theta^2$. A description of the noncommutative conformal structures on noncommutative 2-torus \mathbb{T}_θ^2 is given in [11]. Including a Weyl factor makes the flat geometry on \mathbb{T}_θ^2 into a curved geometry.

We now consider what happens upon dimensional reduction, providing a generalization in the sense that the ansatz (2.20) is replaced with a noncommutative counterpart. Generally, to account for nontriviality of the torus bundle, the ansatz, relating H_3 to F_2 and its dual, is replaced by integration over the fiber. The first operation to deform is the wedge product \wedge , which would be replaced with \wedge_θ . This then leads to a noncommutative version of

dimensional reduction, whereby the resulting equations obtained from the Bianchi identity $dH_3 = 0$ are the same commutative Yang-Mills equations $dF_2 = 0 = d *_4 F_2$.

Note that if, in addition, we deform the Hodge operator $*$ and replace it with their noncommutative counterpart $*_\theta$, then with the new ansatz, the Bianchi identity $dH_3 = 0$ leads to the noncommutative equations $dF_2 = 0$ and $d *_\theta F_2 = 0$, which are the Bianchi identity and equation of motion of noncommutative Yang-Mills theory. It would be interesting to discuss the extension to this most general case, which requires X^4 to be noncommutative as well.

Note that other torus bundles of other fiber dimensions are possible. In fact, one can go all the way and consider the worldvolume as a six-torus. This is considered for example in [13, 22]. We can view this as a limiting case of a 6-torus fiber with a point as a parameter space, and the corresponding deformation is covered by our discussion.

3 The partition function of the model

In this section, as an application, we consider a variation on the M5-brane worldvolume theory as a model for studying S-duality. The main point will be that, to a large extent, we are able to have an analogous discussion on this model of deformed 6-dimensional higher abelian gerbe theory as has been done in the undeformed 4-dimensional abelian Yang-Mills case.

3.1 Evaluation of the partition function

we will outline the evaluation of the partition function. We have found the discussion in [30] and [14] in three and four dimensional abelian Yang-Mills theory, respectively, to be very useful. Our discussion is a generalization to six dimensions and an extension to include deformations. We will start with the rational case and then work globally, making use of the moduli spaces associated to gerbes introduced in [24].

Deformed Hodge theory. In order to proceed, we argue that Hodge theory in general, and the Hodge theorem in particular, continues to hold in the noncommutative setting. Consider the following observations.

1) The cohomology of the deformed de Rham complex does not depend on the deformation, since the differential is the same and the (wedge) product is not used in the definition of cohomology.

2) Since the deformation is isometric, the deformed Laplacian is isospectral to the undeformed one, so in particular there is an isomorphism of the respective zero eigenspaces.

Using these two facts, it follows that Hodge theory in the deformed case is a consequence of standard Hodge theory (in the undeformed case).

Rationally with no topological terms. We start with the theory described by the action (3.14) with no topological term, i.e. with the parameter Θ set to zero. This then reduces to the Yang-Mills gerbe theory described by [24]. Furthermore, we will first work at the level of differential forms, where the cohomology class of the gerbe is trivial. With \mathbb{G} the gauge group, i.e. the group of gauge transformations, the partition function is then given by

$$\begin{aligned} Z(M_\theta^6, e) &= \frac{1}{\text{Vol}(\mathbb{G})} \int_{\Omega_\theta^2(M_\theta^6)} DB e^{\frac{i}{2e^2} \int_{M_\theta^6} H_B \wedge_{\theta} *_{\theta} H_B} \\ &= \frac{1}{\text{Vol}(\mathbb{G})} \int_{\Omega_\theta^2(M_\theta^6)} DB e^{\frac{i}{2e^2} \langle B, d_2 d_2^\dagger B \rangle}, \end{aligned} \quad (3.1)$$

where d_2 is the exterior derivative acting on 2-forms $\Omega_\theta^2(M_\theta^6)$, and d_2^\dagger is the deformed adjoint $*_{\theta} d_2 *_{\theta}$. We need the labeling on the operators in order to distinguish the appearance of various guises of this operator in the partition function. In going from the first line to the second we have used properties of this derivative on deformed differential forms we derived in the previous section.

Decomposing the space as $\Omega_\theta^2(M_\theta^6) = \ker d_2 \oplus (\ker d_2)^\perp$, then we form the space obtained by the action of the gauge group \mathbb{G} as

$$\begin{aligned} [B] &= \mathbb{G} \cdot B \\ &= (\ker d_2) \cdot B \\ &= \{B + B' \mid d_2 B' = 0\}. \end{aligned} \quad (3.2)$$

Generally, one can write the partition function as an integral over the space of fields \mathcal{C} , or as an integral over the moduli space of fields \mathcal{M}_B , which is the space of fields modulo gauge transformations \mathcal{C}/\mathbb{G} . The two are related by the formula

$$Z(M, \lambda) = \frac{1}{\text{Vol}(\mathbb{G})} \int_{\mathcal{C}} DB e^{-i\lambda S(B)} = \frac{1}{\text{Vol}(\mathbb{G})} \int_{\mathcal{C}/\mathbb{G}} D[B] \text{Vol}([B]) e^{-i\lambda S([B])}. \quad (3.3)$$

With this, factoring the volume of ‘gauge group’ we get

$$\begin{aligned} Z(M_\theta^6, e) &= \frac{\text{Vol}(\ker d_2)}{\text{Vol}(\mathbb{G})} \int_{(\ker d_2)^\perp} D[B] e^{\frac{i}{2e^2} \langle B, d_2 d_2^\dagger B \rangle} \\ &= \det' \left(\frac{-i}{4\pi e^2} d_2 d_2^\dagger \right)^{-\frac{1}{2}}. \end{aligned} \quad (3.4)$$

Both terms are infinite, and so a regularization is needed. We will use ζ -function regularization.

Interpretation of the terms. We have local gauge transformations of the B-field $B \mapsto B_2 + dA$, where ⁸ $A_1 \in \Omega^1(M_\theta^6)$. This is a local version of the global transformations in (3.2). The stabilizer of the group of such transformations is flat connections

$$\mathcal{F} = \text{Hom}(\pi_1(M_\theta^6), U(1))/U(1) = \{A \in \Omega^1(M_\theta^6) \mid d_1 A = 0\} , \quad (3.5)$$

where d_1 is the exterior derivative acting on 1-forms. If the fundamental group is abelian then, by the Hurewicz theorem, $\pi_1(M_\theta^6)_{\text{ab}} \cong H_1(M_\theta^6)$. Furthermore, since the conjugation action of $U(1)$ is trivial, then in this case

$$\mathcal{F} = \text{Hom}(H_1(M_\theta^6), \mathbb{R}/\mathbb{Z}) . \quad (3.6)$$

Now the integration can be taken over this quotient space.

In order to choose a representative from each equivalence class $[B]$, we impose a gauge condition. A convenient one is the degree two analog of the Lorentz gauge, i.e. $d_1^\dagger B = 0$. The gauge-fixing condition can be imposed via a delta function. This is the Faddeev-Popov ghost determinant $\det(d_2 d_2^\dagger)$. Now, as the case of other gauge theories, the operator d_1 acting on A itself has zero modes. This implies that we need a ghost-for-ghost determinant, which amounts to dividing by the volume of the stabilizer \mathcal{F} at each point. This in turn needs a gauge-fixing condition, which now in this degree literally looks like the one in abelian Yang-Mills theory, say the Lorentz gauge $d_0^\dagger A = 0$.

Steps in the evaluation of the partition function. We now go through the following steps:

1. We replace $\text{Vol}(\ker d_2)$ by $\text{Vol}(\ker d_1)$.
2. $\text{Vol}(\ker d_1)$ can be evaluated as follows. The projection map $\ker d_i \rightarrow H_{\text{dR}}^i(M_\theta^6)$ induces an isomorphism $\phi_i : \text{Harm}^i(M_\theta^6) \xrightarrow{\cong} H_{\text{dR}}^i(M_\theta^6)$. This follows from the similar isomorphism in the undeformed case. Then, by change of variables, we have $\text{Vol}(\text{Harm}^i(M_\theta^6)) = |\det \phi_i|^{-1} \text{Vol}(H_{\text{dR}}^i(M_\theta^6))$. This then gives that the volume of the stabilizer is $\text{Vol}(\ker d_1) = \det(\phi_1^\dagger \phi_1)^{\frac{1}{2}} \cdot \text{Vol}(\text{Harm}^i(M_\theta^6))$.
3. The ghost-for-ghost determinant $\det(\phi_1^\dagger \phi_1)^{\frac{1}{2}}$ extracts the zero modes from the Faddeev-Popov determinant. This is given by inverse volume of the manifold, as in the classical case (e.g. [30]), i.e $\det(\phi_1^\dagger \phi_1) = (\text{Vol}(M_\theta^6))^{-1}$, since the metric properties are unchanged in the spectral deformation.
4. The complex scaling factor $\frac{-i}{4\pi e^2}$ can be factored out at the expense of introducing the η -function and the ζ -function of the operator $d_2^\dagger d_2$,

$$\det \left(\frac{-i}{4\pi e^2} d_2^\dagger d_2 \right)^{-\frac{1}{2}} = e^{-i\pi\eta(d_2^\dagger d_2)} \left(\frac{-i}{4\pi e^2} \right)^{-\frac{1}{2}\zeta(d_2^\dagger d_2)} \det'(d_2^\dagger d_2)^{-\frac{1}{2}} , \quad (3.7)$$

⁸from here on we will drop the subscript and write Ω for Ω_θ .

where $\eta(P) = \eta(0, P)$ and $\zeta(P) = \zeta(0, P)$ are the analytic continuations of the operator eta-function and zeta-function for a differential operator P of Laplace type. Now the zeta function of our operator is given by $\zeta(d_2^\dagger d_2) = \sum_{j=0}^2 (-1)^j \zeta(\Delta_j)$, where Δ_j is the Hodge Laplacian on j -forms. The zeta function of the latter in turn is given by

$$\zeta(\Delta_p) = -\dim H_{dR}^p(M_\theta^6) + \text{corrections.} \quad (3.8)$$

The first term can be replaced by the corresponding Betti numbers, as follows via the deformed version of the de Rham theorem. The second term will be considered in the next section when investigating modularity.

Overall, with the above contributions, the partition function finally takes the form

$$Z(M_\theta^6, e) = e^{-i\pi\eta(d_2^\dagger d_2)} \left(\frac{-i}{4\pi e^2}\right)^{-\frac{1}{2}(b_2-b_1+b_0)} \frac{\text{Vol}(M_\theta^6)^{\frac{1}{2}}}{\text{Vol}(\text{Harm}^1(M_\theta^6))} \frac{\det'(d_1^\dagger d_1)^{\frac{1}{2}}}{\det'(d_2^\dagger d_2)^{\frac{1}{2}}}, \quad (3.9)$$

up to factors involving asymptotic expansions of the heat kernel.

Adding the purely topological term $\int_{M_\theta^6} H_B \wedge_\theta H_B$ would amount to a numerical correction to the partition function. This is analogous to the instanton term in Yang-Mills theory.

We now recast the above discussion in a more global context. Let \mathcal{G} be an abelian gerbe on M_θ^6 . Isomorphism classes are classified by the Dixmier-Douady class $DD(\mathcal{G}) \in H^3(M_\theta^6; \mathbb{Z})$. Then B can be viewed as a 2-connection with curvature H_B . Fix a gerbe \mathcal{G}_0 over M_θ^6 and a 2-connection B_0 on it such that $H_B(B_0) \in \Omega^3(M_\theta^6)$. The space of 2-connections on \mathcal{G} relative to B_0 is

$$\mathcal{B}(B_0) := \{B \mid B = B_0 + b \text{ with } b \in \Omega^2(M_\theta^6)\}. \quad (3.10)$$

If \mathcal{G}_0 is the trivial gerbe and B_0 is the trivial 2-connection then $\mathcal{B}(B_0) \cong \Omega^2(M_\theta^6)$.⁹ The space $\mathcal{B}(B_0)$ is acted upon by gauge transformations \mathbb{G} and the orbit space of gauge equivalence classes with its quotient topology is $\mathcal{O}(B_0) = \mathcal{B}(B_0)/\mathbb{G}$.

The moduli space of the solutions to the equation of motion of the system can be described explicitly in the classical case [24]. Our current discussion is a straightforward extension to the deformed case.

- Let \mathcal{M}_B denote the moduli space of gauge inequivalent solutions of the deformed equation of motion (1.5) associated to a bundle gerbe \mathcal{G} with connection B over M_θ^6 . Then \mathcal{M}_B is diffeomorphic to the torus $T^{b_2(M)}$ of dimension equal to the second Betti number of M_θ^6 . That is, we have

$$\mathcal{M}_B = H^2(M_\theta^6; \mathbb{R})/H^2(M_\theta^6; \mathbb{Z}) \cong T^{b_2(M)}, \quad (3.11)$$

where the quotient group can be viewed as the gauge group in this case.

⁹Note that in order to impose integrability and introduce topology, we can take $H_3(B_0) \in L^2(M_\theta^6, \Lambda^3 M_\theta^6)$ and then $\mathcal{B}(B_0) \cong L^2(M_\theta^6; \Lambda^6 M_\theta^6)$, analogously to the Yang-Mills case in [14].

- The space \mathcal{B} of all bundle gerbe connections on \mathcal{G} over M_θ^6 is an affine space associated with the vector space $\Omega^1(\mathcal{G})/\pi^*(M_\theta^6)$, where π is the projection from the gerbe to M_θ^6 . Then, as in [24], the total moduli space $\mathcal{M} = \bigcup_{B \in \mathcal{B}} \mathcal{M}_B$ is diffeomorphic to a torus $T^{b_2(M)}$ -bundle over the affine space \mathcal{B} .
- We can now include torsion to account for gerbes with torsion curvatures. These have the virtue of being described by finite-dimensional bundles. We start with the decomposition

$$\begin{aligned} H^2(M_\theta^6; \mathbb{Z}) &\cong \text{Free}(H^2(M_\theta^6; \mathbb{Z})) \times \text{Tor}(H^2(M_\theta^6; \mathbb{Z})) \\ &\cong \mathbb{Z}^{b_2(M)} \times \text{Tor}(H^2(M_\theta^6; \mathbb{Z})) . \end{aligned} \quad (3.12)$$

The the space of gauge equivalence classes of flat gerbes on M_θ^6 is

$$\widehat{\mathcal{M}} \cong T^{b_2(M)} \times \text{Tor}(H^2(M_\theta^6; \mathbb{Z})) . \quad (3.13)$$

We now describe the effect on the corresponding partition function, generalizing the above discussion in the rational case to include contributions from the torsion part. Gerbes with line bundles L such that $c_1(L) \in \text{Tor}(H^2(M_\theta^6; \mathbb{Z}))$ are flat, which implies that the curvature H_B of the corresponding bundle gerbe \mathcal{G} is trivial. Consequently, the action vanishes along these. The contribution will then be simply a numerical factor given by the rank of the torsion part, i.e. $|\text{Tor}(H^2(M_\theta^6; \mathbb{Z}))|$.

The partition function over the full moduli space is

$$\begin{aligned} Z(M_\theta^6, \tau) &= \sum_{[B_0] \in \widehat{\mathcal{M}}} \int_{[B] \in \mathcal{M}} e^{-S(B, \tau)} \\ &= \frac{1}{\text{Vol}(H^2(M_\theta^6; \mathbb{Z}))} \int_{B \in H^2(M_\theta^6; \mathbb{Z})} D[B] e^{-S(B, \tau)} . \end{aligned}$$

By the transformations $H_B \mapsto H_B + db$, we arrive at calculations that are similar to the rational case, but now with b here in place of B there. The remaining discussion, including the effect of the full moduli space and that of torsion, is then a straightforward extension of expression (3.9), which we will leave for the reader to verify.

We have set up the partition function in analogy to the 4-dimensional (abelian) Yang-Mills case. A detailed analysis requires understanding of the geometry and topology of the moduli space $\mathcal{M}_\mathcal{G}$, in particular the renormalized volume and integrals, and also an extensive discussion, adapting that of the seminal work of Verlinde [38], Witten [40], Vafa-Witten [37] to the case of gerbes, going beyond the scope of this letter. Of course we expect that the full evaluation of $Z(M_\theta^6; \tau)$ might lead to connections to geometric and topological invariants of M_θ^6 , as in the commutative case (see [41, 42, 33, 34]).

3.2 Modularity of the partition function

We will consider modularity of the partition function, as an extension to gerbes and to deformed spaces of the Maxwell case in four dimensions. For completeness and ease of comparison we first recall the latter.

The 4d partition function as a modular form. There are two ingredients in the 4-dimensional abelian case [40, 38]. First, the modular transformations of $\text{Im}(\tau)$ are as follows. $\text{Im}(\tau)$ is invariant under a T -transformation, i.e. $\text{Im}(\tau+1) = \text{Im}(\tau)$, and transforms under an S -transformation as $\text{Im}(\frac{-1}{\tau}) = \frac{1}{\tau\bar{\tau}}\text{Im}(\tau)$. The two transformations together imply that $\text{Im}(\tau)$ is a modular form of weights $(-1, -1)$. Second, the Narain-Seigel θ -function of the lattice of signature (b_2^+, b_2^-) is a modular form of weights $(\frac{1}{2}b_2^+, \frac{1}{2}b_2^-)$. Combining the two gives that the partition function of a free abelian gauge theory on a 4-manifold X^4 is a modular form of weights $\frac{1}{4}(\chi - \sigma, \chi + \sigma)$, where $\chi = \chi(X^4)$ and $\sigma = \sigma(X^4)$ are the Euler characteristic and the signature, respectively, of X^4 . These combinations arise because $\chi = 2b_0 - 2b_1 + b_2^+ + b_2^-$ and $\sigma = b_2^+ - b_2^-$.

The general form of the partition function for the abelian gerbe theory. We now go back to our problem and consider the action as in (1.6) but drop the term involving C_6 for simplicity because it does not affect the modularity argument. Assuming as before that 3-curvature is self-adjoint $H_B^\dagger = H_B$ with respect to the deformed inner product, the action is

$$S_\theta(B; \tau) = \frac{1}{2e^2} \int_{M_\theta^6} H_B \wedge_\theta *_\theta H_B + \frac{i\Theta}{2} \int_{M_\theta^6} H_B \wedge_\theta H_B , \quad (3.14)$$

as in the case for analyzing the partition function in the Yang-Mills case. The second term in the action (3.14) is a topological invariant, involving the Dixmier-Douady classes of the underlying gerbe. This then gives that the integrand $e^{-S_\theta(B, \tau)}$ of the partition function is always invariant under the transformation $\Theta \mapsto \Theta + 4\pi$.

For purposes of modularity of the partition function rather than for explicit evaluation of it, one can in principle take the fields in the theory to be either the curvatures or the connections, as is also done in the four-dimensional abelian undeformed case. Choosing curvatures will amount to summing over degree three cohomology classes in place of degree two classes for the connections. As the latter was used explicitly in calculating the partition function in the previous section, for completeness we will also highlight the former in discussing modularity in this section.

The partition function is set as above, and similarly to the Maxwell case in four dimensions, as

$$Z(M_\theta^6; \tau) = \sum_{[\mathcal{G}]} \frac{1}{\text{vol}(\mathcal{M}_\mathcal{G})} \int_{\mathcal{M}_\mathcal{G}} DB e^{-S_\theta(B, \tau)} , \quad (3.15)$$

where the sum is over equivalence classes of gerbes \mathcal{G} on M_θ^6 and the integral is over $\mathcal{M}_\mathcal{G}$, the moduli space of B-fields on \mathcal{G} , with a measure DB .

Using deformed Hodge theory, we proceed as in the abelian Yang-Mills case, by decomposing the B-field as

$$B = B_0 + B_h^{\mathcal{G}} , \quad (3.16)$$

where B_0 is a 2-connection on the trivial gerbe \mathcal{G}_0 which is a global 2-form on M_θ^6 and $B_h^{\mathcal{G}}$ is any 2-connection on the gerbe \mathcal{G} having harmonic curvature $H_h^{\mathcal{G}}$. The partition function then takes the form

$$Z(M_\theta^6; \tau) = \frac{1}{\text{vol}(\mathcal{M}_{\mathcal{G}_0})} \int_{\mathcal{M}_{\mathcal{G}_0}} DB_0 e^{-S_\theta(B_0, \tau)} \sum_{[\mathcal{G}] \in H^3(M_\theta^6; \mathbb{Z})} e^{-S_\theta(B_h^{\mathcal{G}})} . \quad (3.17)$$

We now consider the sum inside the expression for the partition function. On the lattice $H^3(M_\theta^6; \mathbb{Z})$ we have two quadratic forms, both in a sense intrinsically noncommutative; for ω a θ -harmonic (i.e. noncommutative-harmonic) 3-form, these are

$$\begin{aligned} q_1 &:= \int_{M_\theta^6} \omega \wedge_\theta \omega , \\ q_2 &:= \int_{M_\theta^6} \omega \wedge_\theta *_\theta \omega . \end{aligned} \quad (3.18)$$

The first quadratic form q_1 is intrinsically noncommutative, as it vanishes identically in the commutative case, with or without self-duality. The second quadratic form vanishes in the commutative self-dual case, as for the M5-brane theory (without deformation). However, it is also non-identically vanishing in the noncommutative setting, even when self-duality is imposed.

The quadratic form q_1 is indefinite in general with signature, as in the degree two Yang-Mills case, $\sigma(q_1) = (b_3^+, b_3^-)$, where b_3^\pm are the dimensions of the selfdual and ant-selfdual θ -harmonic 3-forms, respectively,

$$\text{Harm}_{\theta}^{3,+} = \{ H_3^\pm \in \Omega^3(M_\theta^6) \mid \Delta_\theta H_3^\pm = 0, H_3^\pm = H_3 \pm *_\theta H_3 \} , \quad (3.19)$$

where Δ_θ is the noncommutative Hodge Laplacian.

We now get back to the expression of the partition function. Taking $\omega = \frac{1}{2\pi} H_3^{\mathcal{G}}$, the sum over gerbes in (3.17) becomes

$$\sum_{\omega \in H^3(M_\theta^6; \mathbb{Z})} \exp \left(-\frac{4\pi}{e^2} q_2 + i \frac{\Theta}{2} q_1 \right) . \quad (3.20)$$

With the identification $\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{e^2}$, this sum is a modular form with holomorphic/anti-holomorphic weights $(\frac{1}{2}b_3^+, \frac{1}{2}b_3^-)$.

Let us consider this more carefully and in more detail. The number of zero modes of the Hodge Laplacian on p -forms coincides with the p -th Betti number b_p . Since we are in an even dimension not divisible by 4, the middle Betti number b_3 is always even. In this case,

the gerbe contributes $b_2 - b_1 + 1$, while the θ -function contributes b_3 . As cancellation is not possible in general in this case, there is a modular anomaly, i.e. the partition function will transform as a modular form of nonzero weights. A priori, and under conditions on lower modes, these weights are expected to be of the form $(\frac{1}{2}b_3^+, \frac{1}{2}b_3^-)$. Furthermore, one can ask whether these can be written in terms of local indices or topological invariants, as was the 4-dimensional case.

Example 1: The six-torus \mathbb{T}^6 . This is an example where all the Betti numbers are present. By the Künneth theorem, the Poincaré polynomial of \mathbb{T}^6 is $P(\mathbb{T}^6) = (1 + x)^6$, from which the Betti numbers are read off as the binomial coefficients, i.e. $b_0 = 1$, $b_1 = 6$, $b_2 = 15$, $b_3 = 20$. Then the Euler characteristic is $\chi(\mathbb{T}^6) = 0$. The gerbe contributes $b_2 - b_1 + 1 = 10$ zero modes, while the θ -function a priori gives $b_3 = 20$. However, taking into account self-duality the latter instead gives a matching $\frac{1}{2}b_3 = 10$. Note that the modularity of the partition function of the M5-brane is demonstrated in [13]. As our deformations are isospectral, the same result holds for the M5-brane with worldvolume the noncommutative space \mathbb{T}_θ^6 .

Example 2: (Product of) Lie groups G . This is an example in which only the third Betti number is relevant. Let G be a compact semi-simple Lie group. Such a group is 2-connected because being semi-simple implies that it is simply connected, and being a Lie group implies that $\pi_2(G) = 0$. So $b_1 = 0 = b_2$. The third Betti number is $b_3 = r$, the number of nonabelian summands in the decomposition of the Lie algebra of G into simple algebras. In this case the gerbe can be taken as the basic gerbe on G whose zero modes are characterized by b_3 . Likewise, the θ -function is characterized by b_3 . Thus, viewing the field as a degree three class, one can arrange for a cancellation, hence obtaining a modular form. The main example in this case is the product of two spheres and quotients thereof. Again, due to the deformation being isospectral the results extend to the noncommutative case, e.g. to $M_\theta^6 = S_\theta^3 \times S_\theta^3$ with the natural torus action.

Modular weights via topological invariants? We would like to investigate this in the noncommutative setting. We have argued above that Hodge theory continues to hold and that, since spectra are preserved, the Betti numbers are also preserved. While our answers to the above question will not be complete, we will provide some arguments in the general case and illustrate with some examples. Furthermore, along the way we will discover a new phenomenon due to noncommutativity.

1. The Euler characteristic. We start by considering effects due to being in six dimension, still considering classical manifolds. The Euler characteristic of M^6 is given by $\chi(M^6) = \sum_{i=0}^6 (-1)^i b_i$ which, by the duality $b_k = b_{6-k}$, becomes $\chi(M^6) = 2b_0 - 2b_1 + 2b_2 - b_3$. Since b_3 is even in six dimensions, then so is the Euler characteristic. This can be simplified if the lower Betti numbers either vanish or cancel each other out. In particular, if M^6 is 2-connected then $\chi(M^6) = 2 - b_3$. It is a useful result of Smale that a 2-connected compact

6-manifold is diffeomorphic to either the 6-sphere S^6 or the connected sum of finite copies of the product $S^3 \times S^3$. For the former case we have $\chi(S^6) = 2$ and for the latter case $\chi(\#_m S^3 \times S^3) = 2 - 2m$, as $b_3 = 2m$ in this case. For $m = 1$ gives $\chi(S^3 \times S^3) = 0$. A main point here is that all of this applies to when M^6 is replaced by its noncommutative deformation M_θ^6 .

2. The signature. Classically, the signature of M^6 is given by $\sigma = \text{sign}(M^6) = b_3^+ - b_3^-$, where b_3^\pm are the eigenvalues of the chirality operator. However, since we are in an even dimension which is not a multiple of 4, i.e. $n = 6 = 2 \cdot 3$ and $3/4 \notin \mathbb{Z}$, we have that the quadratic form

$$q(u, v) = u \cup v = (-1)^{3^2} v \cup u = (-1)^{3^2} q(v, u), \quad u, v \in H^3(M^6; \mathbb{Z}), \quad (3.21)$$

corresponding to the intersection matrix is skew-symmetric. This implies vanishing signature $\sigma = 0$ and so $b_3^+ = b_3^-$. With this, $b_3 = 2b_3^+ = 2b_3^-$. When $M^6 = \#_m(S^3 \times S^3)$, the intersection form is isomorphic to the connected sum of m copies of the standard skew-symmetric hyperbolic form $\mathcal{H}(\mathbb{Z})$ on the lattice $\mathbb{Z} \times \mathbb{Z}$. However, upon deformation, the cup product and in particular the wedge product are no longer skew-symmetric (cf. quadratic forms (3.18)) and so this signals that the signature is not zero. Therefore, one of the byproducts of our analysis is that: *noncommutative deformations also gives us the ability to define the signature operator in dimensions $n = 4k + 2$, which are classically vanishing.* It would be interesting to develop the theory of such signature operators further. We hope to take this up elsewhere.

3.3 Curvature corrections to modular weights

The partition function has a modular anomaly. We have a modular form of weights $(\text{wt}_1, \text{wt}_2)$, i.e. transforms as

$$Z\left(M_\theta^6, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{\text{wt}_1} (c\bar{\tau} + d)^{\text{wt}_2} Z(M_\theta^6, \tau) \quad (3.22)$$

under an $\text{SL}(2, \mathbb{Z})$ transformation.

We will now consider curvature corrections corr_i , $i = 1, 2$ to the modular weights

$$(\text{wt}_1, \text{wt}_2) \mapsto (\text{wt}_1 + \text{corr}_1, \text{wt}_2 + \text{corr}_2), \quad (3.23)$$

which arise from the curvature corrections to the formula (3.8). We will assume that the construction of heat kernels goes through for our isospectral noncommutative deformations. Then, with this assumption and in our case of an even dimension these take the form (see e.g. [32])

$$\frac{1}{16\pi^2} \int_{M_\theta^6} \text{tr}(u_p^6) dV, \quad (3.24)$$

which is part of the expansion of the heat kernel, namely the sections $u_p^n \in C^\infty(M_\theta^6, \text{End}(\Lambda^n M_\theta^6))$, $n = 0, 1, \dots$ appear in the coefficients of the short time asymptotic expansion of the heat kernel of the Laplacians on p -forms

$$\sum_{\lambda \in \text{spec}(\Delta_p)} e^{-\lambda t} \sim \frac{1}{(4\pi t)^2} \sum_{n=0}^{\infty} \left(\int_{M_\theta^6} \text{tr}(u_p^n) dV \right) t^{\frac{n}{2}} \quad \text{as } t \rightarrow 0. \quad (3.25)$$

These coefficients can be calculated in terms of the various curvatures of the manifold, which we assume extend to deformed case (M_θ^6, g) . These can be found explicitly in [19]. Unlike the 4-dimensional case, where these take elegant form in terms of curvature invariants, the 6-dimensional expression is both considerably more complicated and less elegant.

We can consider the example of a 2-torus bundle $\mathbb{T}^2 \rightarrow M^6 \rightarrow X^4$. The fivebrane partition function on the undeformed 6-dimensional worldvolume M^6 has only a \mathbb{Z}_2 symmetry, but the strictly deformed noncommutative manifold $\mathbb{T}_\theta^2 \rightarrow M_\theta^6 \rightarrow X^4$ has a fivebrane partition function with the full $SL(2, \mathbb{Z})$ symmetry.

Overall, we have that our approach via non-commutative deformation evades problems with self-duality in general, and restores modular invariance of the partition function in the model discussed in this section. We hope that main features of these observations will be of use in further explorations and in other settings. We also hope that the noncommutative formulation of the higher abelian gerbe theory on the M5-brane, in the sense discussed in earlier sections, will be useful in further investigations of that theory.

Note that the calculation of the partition function might be more tractable if lifted to eight dimensions, via the Chern-Simons construction, as in [39, 34]. We leave this for future investigation.

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